

NONCOMMUTATIVE QUADRIC SURFACES AND NONCOMMUTATIVE CONIFOLDS

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To the memory of Kentaro Nagao

ABSTRACT. We introduce a compact moduli of noncommutative quadrics, and show that it is the weighted projective space of weight $(2,4,4,6)$. We also introduce a compact moduli of potentials for the conifold quiver, and show that it is the weighted projective space of weight $(1,2,3,4)$. There is a natural morphism from the latter to the former, which is finite of degree 4.

1. INTRODUCTION

The *conifold* is an ordinary double point in dimension 3. It is a cone over $\mathbb{P}^1 \times \mathbb{P}^1$ from the complex point of view, and a cone over $S^2 \times S^3$ from the real point of view. It is the most basic singularity in dimension 3, which appears in various areas of geometry and physics, such as Reid's fantasy [Fri86, Rei87], conifold transition [BCFKvS98], Sasaki-Einstein geometry [Tan79], knot theory [OV00], and AdS/CFT correspondence [KW99].

As a cone over $\mathbb{P}^1 \times \mathbb{P}^1$, the conifold X is obtained by contracting the zero-section of the total space $\tilde{Y} = \text{Spec}_{\mathbb{P}^1 \times \mathbb{P}^1} (\bigoplus_{n=0}^{\infty} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)^{\otimes n})$ of a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$. The contractions along two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ give two crepant resolutions of the conifold;

$$\begin{array}{ccc} & \tilde{Y} & \\ \swarrow & & \searrow \\ Y & & Y^\dagger \\ \searrow & & \swarrow \\ & X & \end{array} .$$

Both Y and Y^\dagger are isomorphic to $\text{Spec}_{\mathbb{P}^1} (\text{Sym}^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)))$. The birational morphism $Y \dashrightarrow Y^\dagger$ obtained from the above diagram, known as the *Atiyah flop* [Ati58], is responsible for the non-separatedness of the moduli space of K3 surfaces.

Besides two crepant resolutions, the conifold has a *noncommutative crepant resolution* [VdB04b, vdB04a]. It is an algebra A_0 of the form $\text{End}_R M$, where $R_0 = \mathbb{C}[x, y, z, w]/(xy - zw)$ is the coordinate ring of X , and $M = R_0 \oplus I$, $I = (x, z) \subset R_0$, is a reflexive R_0 -module. All these crepant resolutions are derived-equivalent [BO, Bri02, VdB04b];

$$(1.1) \quad D(\text{Qcoh } Y) \cong D(\text{Mod } A_0) \cong D(\text{Qcoh } Y^\dagger).$$

The algebra A_0 can be described as the Jacobi algebra $\mathbb{C}Q/(\partial\Phi_0)$ of the quiver in Figure 1.1 with the potential

$$(1.2) \quad \Phi_0 = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

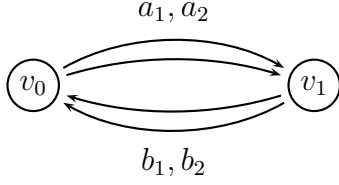


FIGURE 1.1. The conifold quiver

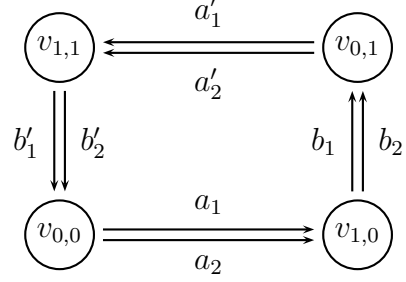


FIGURE 1.2. The double cover

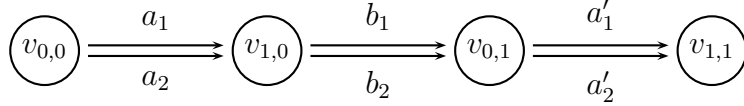


FIGURE 1.3. The $\mathbb{P}^1 \times \mathbb{P}^1$ quiver

Szendrői [Sze08] introduced the notion of *noncommutative Donaldson invariant*, and conjectured that its generating function

$$(1.3) \quad Z(q) = \sum_{v \in \mathbb{Z}^2} D(v) q^v, \quad D(v) = \sum_{n \in \mathbb{Z}} n \chi(\nu^{-1}(n)),$$

has an infinite product expansion

$$(1.4) \quad Z(q) = Z_{\text{DT}}(Y; q_0 q_1, q_1^{-1}) \prod_{m=1}^{\infty} (1 + q_0^m (-q_1)^{m+1})^m.$$

Here $\nu : \mathcal{M}(v) \rightarrow \mathbb{Z}$ is the Behrend function [Beh09] on the moduli space $\mathcal{M}(v)$ of cyclic \tilde{A} -modules of dimension vector $(v_0, v_1, 1)$, and $Z_{\text{DT}}(Y; q_0 q_1, q_1^{-1})$ is the *Donaldson-Thomas invariant* [DT98, Tho00, MNOP06, MNOP06] of the resolved conifold Y . Szendrői's conjecture is proved by Young [You09] in a combinatorial way, and by Nagao and Nakajima [NN11] in an algebro-geometric way. This result is generalized to small toric Calabi-Yau 3-folds in [Nag12, You10], and to motivic invariants in [MMNS12, MN].

In this paper, we consider deformations of the algebra A_0 as a *graded Calabi-Yau algebra of dimension 3*. Since any graded Calabi-Yau algebra of dimension 3 comes from a quiver with potential [Boc08, Theorem 3.1], we study deformation of potentials instead of deformation of algebras. The potential of the conifold quiver in Figure 1.1 is an element of $\text{Sym}^2(V_0 \otimes V_1)$, where V_0 and V_1 are two-dimensional vector spaces, and two potentials give isomorphic graded algebras if they are related by the action of $GL(V_0) \times GL(V_1)$. This motivates us to define the moduli stack of potentials as

$$(1.5) \quad \mathcal{M}_{\text{pot}} = [\text{Sym}^2(V_0 \otimes V_1) / GL(V_0) \times GL(V_1)].$$

The corresponding compact moduli scheme, obtained as the GIT quotient, will be denoted by

$$(1.6) \quad \overline{\mathcal{M}}_{\text{pot}} = \text{Proj} \left(\mathbb{C} [\text{Sym}^2(V_0 \otimes V_1)]^{SL(V_0) \times SL(V_1)} \right).$$

The double cover of the conifold quiver shown in Figure 1.2 inherits the potential from the conifold quiver. When the potential for the conifold quiver is the classical one in (1.2), then the

derived category of modules over the Jacobi algebra of the double cover is equivalent to the derived category of coherent sheaves on the total space of the canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. By taking the quotient with respect to the ideal $\langle b'_1, b'_2 \rangle$ generated by arrows b'_1 and b'_2 , one obtains the quiver in Figure 1.3, equipped with two relations among eight paths from the source vertex $v_{0,0}$ to the sink vertex $v_{1,1}$. This quiver with relations is derived-equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$. This suggests that deformations of the relations of the quiver Figure 1.3 describe noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$. This indeed is the case, and the resulting noncommutative deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ is a noncommutative quadric surface in the sense of Van den Bergh [VdB11].

According to Bondal and Polishchuk [BP93] and Van den Bergh [VdB11], noncommutative quadric surfaces are classified either by

- a quintuple (V_0, V_1, V_2, V_3, W) of 2-dimensional vector spaces V_0, V_1, V_2, V_3 and a one-dimensional subspace $W \subset V_0 \otimes V_1 \otimes V_2 \otimes V_3$, or
- a quadruple $(E, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ of a projective curve E of arithmetic genus one and three line bundles $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ on E satisfying certain conditions.

The moduli stack of quintuples is given by

$$(1.7) \quad \mathcal{M}_{\text{qui}} = [V_0 \otimes V_1 \otimes V_2 \otimes V_3 \otimes W^\vee / GL(V_0) \times GL(V_1) \times GL(V_2) \times GL(V_3) \times GL(W)],$$

and the corresponding GIT quotient is given by

$$(1.8) \quad \overline{\mathcal{M}}_{\text{qui}} = \text{Proj} \left(\mathbb{C}[V_0 \otimes V_1 \otimes V_2 \otimes V_3]^{SL(V_0) \times SL(V_1) \times SL(V_2) \times SL(V_3)} \right).$$

As for quadruples, we give a birational parametrization in terms of a quotient of the fiber product of two copies of certain elliptic surface over the modular curve;

$$(1.9) \quad \overline{\mathcal{M}}_{\text{quad}} = S(2) \times_{X(2)} S(2) / (SL_2(\mathbb{F}_2) \ltimes (\mathbb{Z}/2\mathbb{Z})^4).$$

The main result in this paper is the following:

Theorem 1.1.

- (1) The moduli stack \mathcal{M}_{qui} , after removing the generic stabilizer group \mathbb{G}_m^4 , is birational to the GIT quotient $\overline{\mathcal{M}}_{\text{qui}}$. This in turn is isomorphic to the weighted projective space $\mathbb{P}(2, 4, 4, 6)$.
- (2) The moduli spaces $\overline{\mathcal{M}}_{\text{qui}}$ and $\overline{\mathcal{M}}_{\text{quad}}$ are birational.
- (3) The moduli stack \mathcal{M}_{pot} , after removing the generic stabilizer group \mathbb{G}_m , is birational to the GIT quotient $\overline{\mathcal{M}}_{\text{pot}}$. This in turn is isomorphic to the weighted projective space $\mathbb{P}(1, 2, 3, 4)$.
- (4) There is a natural morphism $\mathcal{M}_{\text{pot}} \rightarrow \mathcal{M}_{\text{qui}}$, which induces a finite morphism $\overline{\mathcal{M}}_{\text{pot}} \rightarrow \overline{\mathcal{M}}_{\text{qui}}$ of degree 4.

The moduli space \mathcal{M}_{qui} also appears in [OU] as the moduli space of 4-qubit states. Theorem 1.1 (1) is a consequence of known results [LT03, Wal08], and has little claim in originality.

This paper is organized as follows: In Section 2, we summarize results from [VdB11] on noncommutative quadric surfaces. Theorem 1.1 (1) is proved in Section 3, and Theorem 1.1 (2) is proved in Section 4. The proofs of Theorems 1.1 (3) and 1.1 (4) are given in Section 5. In Section 6, we introduce the notion of noncommutative conifolds, and discuss their relation with potentials. In Section 7, we show that the motivic Donaldson-Thomas invariants depend on the potential.

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2. NONCOMMUTATIVE QUADRIC SURFACES

See [VdB11, Section 2] for more details of the contents of this section. A \mathbb{Z} -algebra is an algebra with a direct sum decomposition

$$(2.1) \quad A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$$

such that the product satisfies

$$(2.2) \quad A_{ij}A_{kl} = 0 \text{ for } j \neq k, \text{ and}$$

$$(2.3) \quad A_{ij}A_{jk} \subset A_{ik}.$$

We assume that A_{ii} for any $i \in \mathbb{Z}$ has an element e_i , called the *local unit*, which satisfies

$$(2.4) \quad fe_i = f \text{ for any } f \in A_{ki}, \text{ and}$$

$$(2.5) \quad e_ig = g \text{ for any } g \in A_{ij}.$$

A \mathbb{Z} -algebra A is *positively graded* if

$$(2.6) \quad A_{ij} = 0 \text{ for } i > j.$$

A positively graded \mathbb{Z} -algebra A is *connected* if

$$(2.7) \quad \dim A_{ij} < \infty \text{ and } A_{ii} = \mathbb{C}e_i \text{ for any } i, j \in \mathbb{Z}.$$

A *graded A -module* is a right A -module of the form $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that

$$(2.8) \quad M_iA_{kl} = 0 \text{ for } i \neq k, \text{ and}$$

$$(2.9) \quad M_iA_{ij} \subset M_j.$$

A connected \mathbb{Z} -algebra A is *AS-regular* if

$$(2.10) \quad \dim A_{ij} \text{ is bounded by a polynomial in } j - i,$$

$$(2.11) \quad \text{the projective dimension of } S_i \text{ is bounded by a constant independent of } i, \text{ and}$$

$$(2.12) \quad \sum_{j,k} \dim \text{Ext}_{\text{Gr } A}^j(S_k, P_i) = 1 \text{ for any } i \in \mathbb{Z},$$

where $P_i = e_iA$ is a projective A -module, and $S_i = \mathbb{C}e_i$ is a simple A -module. An AS-regular \mathbb{Z} -algebra A is a *3-dimensional cubic AS-regular \mathbb{Z} -algebra* if the minimal resolution of S_i is of the form

$$(2.13) \quad 0 \rightarrow P_{i+4} \rightarrow P_{i+3}^2 \rightarrow P_{i+1}^2 \rightarrow P_i \rightarrow S_i \rightarrow 0.$$

A graded A -module M is *positively bounded* if $M_i = 0$ for sufficiently large i . A graded A -module is a *torsion module* if it can be described as the colimit of a sequence of positively bounded A -modules. The quotient abelian category of the abelian category $\text{Gr } A$ of graded A -modules by the Serre subcategory $\text{Tor } A$ consisting of torsion modules will be denoted by

$$(2.14) \quad \text{Qgr } A = \text{Gr } A / \text{Tor } A.$$

A *noncommutative quadric surface* is an abelian category of the form $\text{Qgr } A$ for a 3-dimensional cubic AS-regular \mathbb{Z} -algebra [VdB11, Definition 3.2].

A 3-dimensional cubic AS-regular \mathbb{Z} -algebra is *linear* if it is of the form

$$(2.15) \quad A = \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-j), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-i))$$

where

$$(2.16) \quad \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(n) = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{\mathbb{P}^1}(k) & n = 2k, \\ \mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{\mathbb{P}^1}(k+1) & n = 2k+1. \end{cases}$$

All other 3-dimensional cubic AS-regular \mathbb{Z} -algebra will be called *elliptic*.

A *quintuple* (V_0, V_1, V_2, V_3, W) consists of

$$(2.17) \quad \text{two-dimensional vector spaces } V_0, V_1, V_2, V_3, \text{ and}$$

$$(2.18) \quad \text{a one-dimensional subspace } W \text{ of } V_0 \otimes V_1 \otimes V_2 \otimes V_3.$$

A quintuple is *geometric* if a basis w of W satisfies

$$(2.19) \quad \langle \phi_j \otimes \phi_{j+1}, w \rangle \neq 0 \in V_{j+2} \otimes V_{j+3}$$

for any $j = 0, 1, 2, 3$ and any non-zero $\phi_j \in V_j^\vee$, $\phi_{j+1} \in V_{j+1}^\vee$, where indices are taken modulo 4 [VdB11, Definition 4.7]. A geometric quintuple is *linear* if it can be written as

$$(2.20) \quad w = x_0 x_1 y_2 y_3 - y_0 x_1 x_2 y_3 - x_0 y_1 y_2 x_3 + y_0 y_1 x_2 x_3$$

for a suitable choice of a basis $\{x_i, y_i\}$ of V_i for $i = 0, 1, 2, 3$ (see [VdB11, Lemma 4.6]). All other geometric quintuples are called *elliptic*. Two quintuples (V_0, V_1, V_2, V_3, W) and $(V'_0, V'_1, V'_2, V'_3, W')$ are isomorphic if there are linear isomorphisms $V_i \cong V'_i$ for $i = 0, 1, 2, 3$ sending $W \subset V_0 \otimes V_1 \otimes V_2 \otimes V_3$ to $W' \subset V'_0 \otimes V'_1 \otimes V'_2 \otimes V'_3$.

A globally generated line bundle \mathcal{L} on a scheme C defines a morphism $\phi_{\mathcal{L}} : C \rightarrow \mathbb{P}(H^0(\mathcal{L})^\vee)$, whose direct product will be denoted by

$$(2.21) \quad \phi_{\mathcal{L}, \mathcal{L}'} : C \rightarrow \mathbb{P}(H^0(\mathcal{L})^\vee) \times \mathbb{P}(H^0(\mathcal{L}')^\vee).$$

An *admissible quadruple* $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ consists of a curve C and line bundles $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ on C of degree 2 such that

$$(2.22) \quad h^0(\mathcal{L}_i) = 2 \text{ for } i = 0, 1, 2,$$

$$(2.23) \quad C \text{ is embedded as a divisor of bidegree } (2, 2) \text{ in } \mathbb{P}^1 \times \mathbb{P}^1 \text{ by both } \phi_{\mathcal{L}_0, \mathcal{L}_1} \text{ and } \phi_{\mathcal{L}_1, \mathcal{L}_2},$$

$$(2.24) \quad \deg(\mathcal{L}_0|_E) = \deg(\mathcal{L}_2|_E) \text{ for every irreducible component } E \text{ of } C, \text{ and}$$

$$(2.25) \quad \mathcal{L}_0 \not\cong \mathcal{L}_2.$$

Two admissible quadruples $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ and $(C', \mathcal{L}'_0, \mathcal{L}'_1, \mathcal{L}'_2)$ are *isomorphic* if there is an isomorphism $\varphi : C \rightarrow C'$ of curves such that $\varphi^* \mathcal{L}'_i \cong \mathcal{L}_i$ for $i = 0, 1, 2$.

There is a bijective correspondence among isomorphisms classes of

$$(2.26) \quad \text{elliptic 3-dimensional cubic AS-regular } \mathbb{Z}\text{-algebras,}$$

$$(2.27) \quad \text{elliptic quintuples, and}$$

$$(2.28) \quad \text{admissible quadruples,}$$

given as follows:

- [VdB11, §4.2] : For a 3-dimensional cubic AS-regular \mathbb{Z} -algebra $A = \bigoplus_{i \leq j} A_{ij}$, set $V_i = A_{i, i+1}$ for $i = 0, 1, 2, 3$ and $R_i = \ker(V_i \otimes V_{i+1} \otimes V_{i+2} \rightarrow A_{i, i+3})$ for $i = 0, 1$. Then $W_0 = R_0 \otimes V_3 \cap V_0 \otimes R_1$ is one-dimensional, and $(V_0, V_1, V_2, V_3, W_0)$ gives a geometric quintuple.
- [VdB11, §4.4] : For a geometric quintuple $Q = (V_0, V_1, V_2, V_3, W_0 = kw)$, choose bases $\{x_i, y_i\}$ of V_i and write $w = fx_3 + gy_3$. Let $C \subset \mathbb{P}(V_0) \times \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ be the variety defined by the equations $\{f, g\}$, and set $\mathcal{L}_i = p_i^*(\mathcal{O}(1))$ for $i = 0, 1, 2$, where $p_i : C \rightarrow \mathbb{P}(V_i)$ are the

projections. The geometricity of Q implies that $\phi_{\mathcal{L}_0, \mathcal{L}_1} = (p_0, p_1)$ and $\phi_{\mathcal{L}_1, \mathcal{L}_2} = (p_1, p_2)$ are closed embeddings, and one obtains a quadruple $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$.

- [VdB11, §4.5] : For an admissible quadruple $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$, construct a helix $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ by

$$\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{-1} \otimes \mathcal{L}_{i+2}^{-1} \otimes \mathcal{L}_{i+3} = \mathcal{O}_C.$$

Then the \mathbb{Z} -algebra generated by $A_{i,i+1} = V_i = H^0(C, \mathcal{L}_i)$ with the relations $R_i = \ker(V_i \otimes V_{i+1} \otimes V_{i+2} \rightarrow A_{i,i+3})$ is an elliptic 3-dimensional cubic AS-regular \mathbb{Z} -algebra.

3. MODULI OF QUINTUPLES

The invariant ring of $\mathbb{C}[V_0 \otimes V_1 \otimes V_2 \otimes V_3]$ with respect to the action of $SL(V_0) \times SL(V_1) \times SL(V_2) \times SL(V_3)$ is given in [LT03]. In this section, we give a description of this invariant ring along the lines of [VDDMV02, Wal08] and prove Theorem 1.1.1.

Let

$$(3.1) \quad \omega(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, y_1 \mathbf{e}_1 + y_2 \mathbf{e}_2) = x_1 y_2 - x_2 y_1$$

be a skew-symmetric bilinear form on \mathbb{C}^2 . The action of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by

$$(3.2) \quad (g, h) \cdot u \otimes v = (gu) \otimes (hv).$$

preserves the non-degenerate symmetric bilinear form

$$(3.3) \quad (v \otimes w, x \otimes y) = \omega(v, x) \omega(w, y),$$

whose Gram matrix with respect to the basis $\{\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2\}$ is given by

$$(3.4) \quad J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

An orthonormal basis with respect to J is given by

$$(3.5) \quad T := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} & 0 \\ 0 & -1 & 1 & 0 \\ \sqrt{-1} & 0 & 0 & -\sqrt{-1} \end{pmatrix},$$

so that one obtains a homomorphism

$$(3.6) \quad \begin{array}{ccc} \varphi : & SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) & \rightarrow & SO_4(\mathbb{C}) \\ & \downarrow \Psi & & \downarrow \Psi \\ & (g_1, g_2) & \mapsto & T^T(g_1 \otimes g_2)T \end{array}$$

of algebraic groups. The induced morphism

$$(3.7) \quad (d\varphi)_e : \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{so}(\mathbb{C})$$

of Lie algebras is an isomorphism called the *accidental isomorphism*. Hence the morphism φ is étale surjective, and one can easily see that

$$(3.8) \quad \text{Ker } \varphi = \{\pm(I, I)\} \in SL_2(\mathbb{C}) \times SL_2(\mathbb{C}).$$

One can identify the ring $\mathbb{C}[V_0 \otimes V_1 \otimes V_2 \otimes V_3]$ with the ring $\mathbb{C}[U \otimes U']$ equipped with the action of $SO(U) \times SO(U')$ through the homomorphisms $SL(V_0) \times SL(V_1) \rightarrow SO(U)$ and $SL(V_2) \times SL(V_3) \rightarrow SO(U')$, in such a way that the invariant rings are isomorphic;

$$(3.9) \quad \mathbb{C}[V_0 \otimes V_1 \otimes V_2 \otimes V_3]^{SL(V_0) \times SL(V_1) \times SL(V_2) \times SL(V_3)} \cong \mathbb{C}[U \otimes U']^{SO(U) \times SO(U')}.$$

The action of $SO(U) \times SO(U')$ on $U \otimes U'$ can be identified with the action of $SO(U) \times SO(U)$ on $\text{End } U$ given by

$$(3.10) \quad (g, h) \cdot X = gXh^T.$$

Singular value decomposition allows us to turn any $n \times n$ matrix into a diagonal matrix under this action. The entries x_1, \dots, x_n of the resulting diagonal matrix is unique up to

- permutations of x_1, \dots, x_n , and
- sign changes $x_i \mapsto \epsilon_i x_i$, where $\epsilon_i = \pm 1$ and $\epsilon_1 \cdots \epsilon_n = 1$.

As a consequence, one has an isomorphism

$$(3.11) \quad \mathbb{C}[\text{End } U]^{SO(U) \times SO(U)} \cong \mathbb{C}[x_1, x_2, x_3, x_4]^W$$

$$(3.12) \quad = \mathbb{C}[f_2, f_4, g_4, f_6],$$

where $W = \mathfrak{S}_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$ acts on \mathbb{C}^4 by permutations and sign changes, and

$$(3.13) \quad f_{2d} = \sum_{i=1}^4 x_i^{2d},$$

$$(3.14) \quad g_4 = x_1 x_2 x_3 x_4.$$

The corresponding elements of $\mathbb{C}[\text{End } U]^{SO(U) \times SO(U)}$ are

$$(3.15) \quad f_{2d}(X) = \text{Tr}((X^T X)^d),$$

$$(3.16) \quad g_4 = \det(X).$$

It follows that

$$(3.17) \quad \text{an element } X \in \text{End } U \text{ is stable if and only if } g_4(X) \neq 0,$$

$$(3.18) \quad \text{an element } X \in \text{End } U \text{ is unstable if and only if } X^T X \text{ is nilpotent, and}$$

$$(3.19) \quad \text{the GIT quotient is given by } \text{Proj } \mathbb{C}[f_2, f_4, g_4, f_6] = \mathbb{P}(2, 4, 4, 6).$$

4. ELLIPTIC QUADRUPLES

The principal congruence subgroup

$$(4.1) \quad \Gamma(2) = \text{Ker}(SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{F}_2))$$

of the modular group $SL_2(\mathbb{Z})$ acts on the upper half plane \mathbb{H} by

$$(4.2) \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}.$$

The generic stabilizer of this action is $\langle \pm I_2 \rangle$, and the induced action of $\Gamma(2)/\langle \pm I_2 \rangle$ is free. The quotient $X'(2) = \mathbb{H}/\Gamma(2)$ can be compactified to the modular curve by adding three cusps. The modular curve $X(2) \cong \text{Proj } \mathbb{C}[\lambda_0, \lambda_1]$ is known as the λ -line, which parameterizes the family

$$(4.3) \quad E_\lambda = \{[X : Y : Z] \in \mathbb{P}(1, 1, 2) \mid Z^2 = XY(X - Y)(\lambda_1 X - \lambda_0 Y)\}$$

of elliptic curves equipped with level-2 structures.

Consider the action of the group $\Gamma(2) \ltimes \mathbb{Z}^2$ on $\mathbb{H} \times \mathbb{C}$ defined by

$$(4.4) \quad (\gamma, m, n) : (\tau, z) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z + m\tau + n}{c\tau + d} \right).$$

The action of the subgroup \mathbb{Z}^2 is free, and the action of the quotient group $\Gamma(2)$ on $\mathbb{H} \times \mathbb{C}/\mathbb{Z}^2$ is free outside of the four sections corresponding to the four 2-division points. Let $S'(2)$ be the

branched double cover of the quotient $\mathbb{H} \times \mathbb{C}/\Gamma(2) \ltimes \mathbb{Z}^2$ along the images of those four sections. The natural projection

$$(4.5) \quad S'(2) \rightarrow X'(2), \quad (\tau, z) \mapsto \tau$$

is an elliptic fibration, which can be identified with the family

$$(4.6) \quad V(Z^2 - XY(X - Y)(\lambda_1 X - \lambda_0 Y)) \subset \mathbb{P}(1, 1, 2)_{X:Y:Z} \times \mathbb{A}_\lambda^1 \rightarrow \mathbb{A}_\lambda^1.$$

The origin of the elliptic curve is given by $[1 : 0 : 0]$, and other 2-torsion points are given by $[1 : 1 : 0]$, $[\lambda_0 : \lambda_1 : 0]$, and $[0 : 1 : 0]$.

Let $\overline{S}(2)$ be the natural compactification

$$(4.7) \quad V(Z^2 - XY(X - Y)(\lambda_1 X - \lambda_0 Y)) \subset \mathbb{P}(1, 1, 2)_{X:Y:Z} \times \mathbb{P}_\lambda^1$$

of $S'(2)$. The natural morphism to $\mathbb{P}_\lambda^1 \cong X(2)$ has three singular fibers of type I_1 at the cusps $\lambda = 0, 1, \infty$. The surface $\overline{S}(2)$ itself has three A_1 -singularities at the singular points of the singular fibers, and we write its minimal resolution as $S(2)$.

Proposition 4.1. *The elliptic fibration $S(2) \rightarrow X(2)$ has three singular fibers of type I_2 at $\lambda = 0, 1, \infty$, and the natural action of $SL(2, \mathbb{F}_2) \ltimes (\mathbb{Z}/2\mathbb{Z})^2$ on $S'(2)$ extends to $S(2)$.*

Proof. In the coordinates of $\overline{S}(2)$, the three translations by the points $(1 : 1 : 0)$, $(\lambda_0 : \lambda_1 : 0)$, $(0 : 1 : 0)$ of order two can be written respectively as follows:

$$(4.8) \quad (X : Y : Z) \mapsto (\lambda(X - Y) : X - \lambda Y : \lambda(\lambda - 1)Z)$$

$$(4.9) \quad (X : Y : Z) \mapsto (-X + \lambda Y : -X + Y : (\lambda - 1)Z)$$

$$(4.10) \quad (X : Y : Z) \mapsto (\lambda Y : X : \lambda Z)$$

By direct calculations, we can check that these are merely birational maps on $\overline{S}(2)$ but are genuine automorphisms on $S(2)$. \square

Consider the three-fold $S(2) \times_{X(2)} S(2)$, which is a compactification of the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -covering of the quotient $\mathbb{H} \times \mathbb{C} \times \mathbb{C}/\Gamma(2) \ltimes \mathbb{Z}^4$ by the action

$$(4.11) \quad (\gamma, m_1, n_1, m_2, n_2) : (\tau, z_1, z_2) \mapsto \left(\gamma(\tau), \frac{z_1 + m_1\tau + n_1}{c\tau + d}, \frac{z_2 + m_2\tau + n_2}{c\tau + d} \right).$$

To a point $p = (\tau, z_1, z_2) \in S'(2) \times_{X'(2)} S'(2)$, one can associate the quadruple

$$(4.12) \quad (E_\tau, \mathcal{O}(2o), \mathcal{O}(2z_1), \mathcal{O}(2z_2)),$$

where $o = [0] \in E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ is the origin. The quadruple (4.12) is admissible if $2z_2 \not\sim 2o$. Two elliptic quadruples $(E_\tau, \mathcal{O}(2o), \mathcal{O}(2z_1), \mathcal{O}(2z_2))$ and $(E_{\tau'}, \mathcal{O}(2o), \mathcal{O}(2z'_1), \mathcal{O}(2z'_2))$ are isomorphic if

- (1) the elliptic curves E_τ and $E_{\tau'}$ are isomorphic, i.e., $\tau' = \gamma(\tau)$ for some $\gamma \in SL_2(\mathbb{Z})$, and
- (2) $2z_i \sim 2z'_i$ for $i = 1, 2$, i.e., z_i and z'_i are related by 2-torsion translation.

Together, they form the group

$$(4.13) \quad G = SL_2(\mathbb{F}_2) \ltimes (\mathbb{Z}/2\mathbb{Z})^4$$

acting on $S'(2) \times_{X'(2)} S'(2)$, and one obtains the following:

Proposition 4.2. *Two points $p, q \in S'(2) \times_{X'(2)} S'(2)$ belong to the same orbit of the action of G if and only if the associated quadruples are isomorphic.*

Theorem 1.1 (2) follows from Proposition 4.2 and the equivalence between (2.27) and (2.28).

5. POTENTIALS

One can use the homomorphism (3.6) to identify the two rings $\mathbb{C} [\mathrm{Sym}^2(V_0 \otimes V_1)]$ and $\mathbb{C} [\mathrm{Sym}^2(U)]$ in such a way that the invariant subrings are isomorphic;

$$(5.1) \quad \mathbb{C} [\mathrm{Sym}^2(V_0 \otimes V_1)]^{SL(V_0) \times SL(V_1)} \cong \mathbb{C} [\mathrm{Sym}^2(U)]^{SO(U)}.$$

Any symmetric matrix $X \in \mathrm{Sym}^2(U)$ can be diagonalized by the conjugate action

$$(5.2) \quad g \cdot X = gXg^T$$

of $SO(U)$. The entries x_1, \dots, x_4 of the resulting diagonal matrix is unique up to permutations. As a consequence, one has an isomorphism

$$(5.3) \quad \mathbb{C} [\mathrm{Sym}^2(U)]^{SO(U)} \cong \mathbb{C}[x_1, x_2, x_3, x_4]^{\mathfrak{S}_4} = \mathbb{C}[f_1, f_2, f_3, f_4],$$

where

$$(5.4) \quad f_d = \sum_{i=1}^4 x_i^d.$$

The corresponding elements of $\mathbb{C} [\mathrm{End} U]^{SO(U)}$ are

$$(5.5) \quad f_d(X) = \mathrm{tr}(X^d).$$

It follows that

$$(5.6) \quad \text{an element } X \in \mathrm{Sym}^2(U) \text{ is unstable if and only if } X \text{ is nilpotent, and}$$

$$(5.7) \quad \text{the GIT quotient is given by } \mathrm{Proj} \mathbb{C}[f_1, f_2, f_3, f_4] = \mathbb{P}(1, 2, 3, 4).$$

With a potential $\Phi \in \mathrm{Sym}^2(V_0 \otimes V_1) \subset (V_0 \otimes V_1) \otimes (V_0 \otimes V_1)$, one can associate the quintuple $(V_0, V_1, V_0, V_1, \mathbb{C}\Phi)$. This gives a morphism

$$(5.8) \quad F: \mathcal{M}_{\mathrm{pot}} \rightarrow \mathcal{M}_{\mathrm{qui}}$$

of stacks, which induces a rational map of the corresponding GIT quotients. This rational map can be identified with the natural projection from $\overline{\mathcal{M}}_{\mathrm{pot}} \cong \mathbb{P}^3/\mathfrak{S}_3$ to $\overline{\mathcal{M}}_{\mathrm{pot}} \cong \mathbb{P}^3/(\mathfrak{S}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3)$, and hence a posteriori is a morphism. Since the action $[x_1 : x_2 : x_3 : x_4] \mapsto [-x_1 : -x_2 : -x_3 : -x_4]$ is trivial on \mathbb{P}^3 , the degree of this morphism coincides with the cardinality of the group $(\mathbb{Z}/2\mathbb{Z})^3/(\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, which is four. The invariants g_4 and f_6 can be described in $\mathbb{C}[f_1, f_2, f_3, f_4]$ as

$$(5.9) \quad g_4 = \frac{f_1^4}{24} - \frac{f_1^2 f_2}{4} + \frac{f_1 f_3}{3} + \frac{f_2^2}{8} - \frac{f_4}{4},$$

$$(5.10) \quad f_6 = -\frac{f_1^6}{24} + \frac{3f_1^4 f_2}{8} - \frac{2f_1^3 f_3}{3} - \frac{3f_1^2 f_2^2}{8} + \frac{3f_1^2 f_4}{4} - \frac{f_2^3}{8} + \frac{3f_2 f_4}{4} + \frac{f_3^2}{3}.$$

6. NONCOMMUTATIVE CONIFOLDS

A *quiver* (Q_0, Q_1, s, t) consists of a set Q_0 of vertices, a set Q_1 of arrows, and two maps $s, t : Q_1 \rightarrow Q_0$ from Q_1 to Q_0 . The vertices $s(a)$ and $t(a)$ are called the *source* and the *target* of the arrow a . A *path* on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \dots, a_1)$ such that $s(a_{k+1}) = t(a_k)$ for $k = 1, \dots, n-1$. We also allow for a path e_i of length zero, starting and ending at the same

vertex $i \in Q_0$. The *path algebra* $\mathbb{C}Q$ of a quiver Q is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths;

$$(b_m, \dots, b_1) \cdot (a_n, \dots, a_1) = \begin{cases} (b_m, \dots, b_1, a_n, \dots, a_1) & s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A *cyclic path* is a path (a_n, \dots, a_1) such that $s(a_1) = t(a_n)$. There is a cyclic group action $(a_n, \dots, a_1) \mapsto (a_{n-1}, \dots, a_1, a_n)$ on the set of cyclic paths. A *potential* is an element Φ of the invariant subspace $(\mathbb{C}Q)^{\text{cycl}}$ of the subspace of $\mathbb{C}Q$ spanned by cyclic paths. The partial derivative of a path by an arrow a is defined by

$$(6.1) \quad \frac{\partial(a_n, \dots, a_1)}{\partial b} = \begin{cases} (a_{n-1}, \dots, a_1) & a_n = b, \\ 0 & \text{otherwise.} \end{cases}$$

The *Jacobi algebra* of a quiver (Q, Φ) with potential is the quotient $\mathbb{C}Q/(\partial\Phi)$ of the path algebra $\mathbb{C}Q$ by the two-sided ideal

$$(6.2) \quad (\partial\Phi) = \left(\frac{\partial\Phi}{\partial a} \right)_{a \in Q_1}$$

generated by the partial derivatives. The path algebra of a quiver has a natural grading coming from the length of paths. The Jacobi algebra $\mathbb{C}Q/(\partial\Phi)$ inherits this grading if the potential Φ is homogeneous.

An algebra A is *homologically smooth* if A has finite projective dimension as an A -bimodule (i.e. as a module over $A^e = A \otimes_{\mathbb{C}} A^{\text{op}}$). A homologically smooth algebra is a *Calabi-Yau algebra of dimension 3* if there is an isomorphism $f: A \xrightarrow{\sim} A^![3]$ such that $f = f^![3]$, where $A^! = \mathbb{R}\text{Hom}_{A^e}(A, A^e)$ is the inverse of the rigid dualizing complex [Gin06, Definition 3.2.3].

Let A be a flat deformation of the noncommutative crepant resolution $A_0 = \mathbb{C}Q/(\partial\Phi_0)$ of the conifold, where Q is the quiver in Figure 1.1 and Φ_0 is the potential in (1.2). We assume that A is a graded Calabi-Yau algebra of dimension 3, so that it can be described as the Jacobi algebra $\mathbb{C}Q/(\partial\Phi)$ of a quiver (Q, Φ) with potential by [Boc08, Theorem 3.1]. The flatness of the deformation implies the invariance of the Hilbert series of the graded ring. It follows that the degree of the potential Φ must be preserved under deformation.

Let \tilde{Q} be the quiver in Figure 1.2 obtained as the double cover of the quiver Q . The pull-back of the potential Φ by the natural homomorphism $\mathbb{C}\tilde{Q} \rightarrow \mathbb{C}Q$ gives a potential $\tilde{\Phi}$ of the quiver \tilde{Q} . The quotient of the Jacobi algebra $\mathbb{C}\tilde{Q}/(\partial\tilde{\Phi})$ by the two-sided ideal generated by b'_1 and b'_2 gives the path algebra of the quiver in Figure 1.3 with two relations among eight paths from $v_{0,0}$ to $v_{1,1}$. These two relations can be regarded either as a two-dimensional subspace of $V_0 \otimes V_1 \otimes V_0$ or a one-dimensional subspace of $V_0 \otimes V_1 \otimes V_0 \otimes V_1$. The latter coincides with the $V_0 \otimes V_1 \otimes V_0 \otimes V_1$ component of the potential $\tilde{\Phi}$, which we write as Φ by abuse of notation. In this way, one obtains a quintuple $(V_0, V_1, V_0, V_1, \mathbb{C}\Phi)$ from a potential Φ . The potential Φ is said to be *geometric* if the associated quintuple is geometric.

It follows from the construction of the \mathbb{Z} -algebra $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ from a quintuple that the Jacobi algebra $\mathbb{C}Q/(\partial\Phi)$ is related to the \mathbb{Z} -algebra by

$$(6.3) \quad e_i \mathbb{C}Q/(\partial\Phi) e_j = \begin{cases} \bigoplus_{n=0}^{\infty} A_{0,2n} & i = j, \\ \bigoplus_{n=0}^{\infty} A_{0,2n+1} & i \neq j. \end{cases}$$

The flatness of the deformation is equivalent to the geometricity of the potential [VdB11, Lemma 4.9]. Geometricity of Φ implies that the associated \mathbb{Z} -algebra A is AS-regular of dimension 3 [VdB11, Theorem 4.31], which in turn implies that $\mathbb{C}Q/(\partial\Phi)$ is Calabi-Yau of dimension 3. The

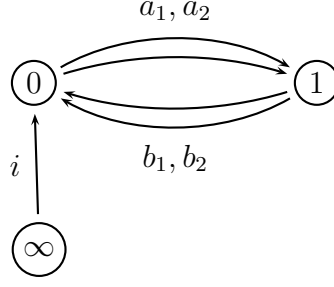


FIGURE 7.1. The framed conifold quiver

subring $R = e_0 \mathbb{C}Q / (\partial\Phi)e_0$ of the Jacobi algebra of a geometric potential will be called a *noncommutative conifold*.

A *categorical resolution* [KL, Definition 1.3] (cf. also [Kuz08, Lun10]) of a ring R is a smooth cocomplete compactly generated triangulated category \mathcal{T} and an adjoint pair $\pi^* : D(R) \rightleftharpoons \mathcal{T} : \pi_*$ of functors such that

- (1) the natural morphism $\text{id}_{D(R)} \rightarrow \pi_* \pi^*$ of functors is an isomorphism,
- (2) both π^* and π_* commute with arbitrary direct sum, and
- (3) $\pi_*(\mathcal{T}^c) \subset D^b \text{mod}(R)$.

When R is the subring $e_0 \mathbb{C}Q / (\partial\Phi)e_0$ of the Jacobi algebra $\mathbb{C}Q / (\partial\Phi)$ of the conifold quiver with a geometric potential Φ , the adjoint pair $\pi^* \vdash \pi_*$ of the functors

$$(6.4) \quad \pi_* : D(\mathbb{C}Q / (\partial\Phi)) \rightarrow D(R), \quad M \mapsto Me_0$$

and

$$(6.5) \quad \pi^* : D(R) \rightarrow D(\mathbb{C}Q / (\partial\Phi)), \quad N \mapsto N \otimes_R e_0 \mathbb{C}Q / (\partial\Phi)$$

is a categorical resolution of the noncommutative conifold R .

7. DONALDSON-THOMAS INVARIANTS

Let \tilde{Q} be the quiver in Figure 7.1, equipped with the potential $\tilde{\Phi}$ inherited from a potential Φ of the conifold quiver Q . A *framed representation* of the conifold quiver is a module over the Jacobi algebra $\mathbb{C}\tilde{Q} / (\partial\tilde{\Phi})$, i.e., a collection $M = ((V_0, V_1, V_\infty), (A_1, A_2, B_1, B_2, I))$ of vector spaces (V_0, V_1, V_∞) and maps

$$(7.1) \quad A_i : V_0 \rightarrow V_1, \quad i = 1, 2,$$

$$(7.2) \quad B_i : V_1 \rightarrow V_0, \quad i = 1, 2,$$

$$(7.3) \quad I : V_\infty \rightarrow V_0$$

satisfying the relation $(\partial\tilde{\Phi})$. The *dimension vector* of M is given by

$$(7.4) \quad v = (v_0, v_1, v_\infty) = (\dim V_0, \dim V_1, \dim V_\infty).$$

A *stability parameter* is an element $\theta = (\theta_0, \theta_1, \theta_\infty)$ of \mathbb{R}^3 . The *slope* of a representation $M = ((V_0, V_1, V_\infty), (A_1, A_2, B_1, B_2, I))$ with respect to the stability parameter θ is defined by

$$(7.5) \quad \theta(M) = \frac{\theta_0 \dim V_0 + \theta_1 \dim V_1 + \theta_\infty \dim V_\infty}{\dim V_0 + \dim V_1 + \dim V_\infty}.$$

A representation M is θ -stable if one has

$$(7.6) \quad \theta(M') < \theta(M).$$

for any non-zero proper submodule $M' \subset M$. The moduli space of framed θ -stable representations with dimension v will be denoted by $\mathcal{M}_\theta(v)$. The *numerical DT invariant* is defined as the weighted Euler characteristic (1.3) with respect to the Behrend function on $\mathcal{M}_\theta(v)$. We expect that the numerical DT invariant is independent of the potential Φ .

The numerical DT invariant has a motivic refinement, which is defined in terms of *virtual motives* [BBS13]. When the potential is the one given in (1.2), then $\mathcal{M}_\theta(v)$ for $v = (1, 1, 1)$ and $\theta = (-1, -1, 2)$ is the resolved conifold

$$(7.7) \quad Y = [(\mathbb{A}^4 \setminus \{(0, 0)\} \times \mathbb{A}^2) / \mathbb{C}^\times] \cong \text{Spec}_{\mathbb{C}\mathbb{P}^1} (\text{Sym}^* (\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))),$$

and the virtual motive is given by

$$(7.8) \quad [Y]_{\text{vir}} = \mathbb{L}^{-3/2} [Y],$$

where \mathbb{L} is the Lefschetz motive. When the potential is given by

$$(7.9) \quad \Phi = \alpha a_1 b_1 a_1 b_1 + \beta a_1 b_2 a_1 b_2 + \gamma a_2 b_1 a_2 b_1 + \delta a_2 b_2 a_2 b_2,$$

for general α, β, γ and δ , the moduli space $\mathcal{M}_\theta(v)$ is the subscheme of Y defined by

$$(7.10) \quad a_1(\alpha b_1^2 + \beta b_2^2) = 0,$$

$$(7.11) \quad a_2(\gamma b_1^2 + \delta b_2^2) = 0,$$

$$(7.12) \quad b_1(\alpha a_1^2 + \beta a_2^2) = 0,$$

$$(7.13) \quad b_2(\gamma a_1^2 + \delta a_2^2) = 0.$$

The motivic DT invariant in this case is given by

$$(7.14) \quad -\mathbb{L}^{-3/2} ([\Phi^{-1}(1)] - [\Phi^{-1}(0)])$$

by [BBS13, Theorem B.1], where Φ is considered as a map from Y to \mathbb{A}^1 . Being the difference of two subschemes, this motive is clearly distinct from (7.8).

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